A NOTE ON THE COMPLETENESS OF $\mathcal{C}_c(X, Y)$

JAN HARM VAN DER WALT

Abstract. It is known that there are complete, Hausdorff and regular convergence vector spaces X and Y such that $\mathcal{L}_c(X, Y)$, the space of continuous linear mappings from X into Y equipped with the continuous convergence structure, is not complete. In this paper, we give sufficient conditions on a convergence vector space Y such that $\mathcal{C}_c(X, Y)$ is complete for any convergence space X. In particular, we show that this is true for every complete and Hausdorff topological vector space Y.

1. Introduction

It is well known [3] that $\mathcal{C}_c(X)$, the space of continuous, scalar-valued functions on a convergence space X equipped with the continuous convergence structure, is a complete convergence vector space. An immediate consequence of this fact, see [2], is that the continuous dual $\mathcal{L}_cX$ of a convergence vector space X is complete. On the other hand, Butzmann [5] gave an example of Hausdorff, regular and complete convergence vector spaces X and Y such that the convergence vector space $\mathcal{L}_c(X, Y)$ is not complete. Here, as is standard in the literature, see for instance [2], we denote by $\mathcal{L}(X, Y)$ the vector space of continuous linear mappings from X into Y, and $\mathcal{L}_c(X, Y)$ denotes this space equipped with the continuous convergence structure.

In this paper, we show that if Y is complete, Hausdorff and topological, then $\mathcal{C}_c(X, Y)$ is complete for every convergence space X. An immediate consequence is that $\mathcal{L}_c(X, Y)$ is complete whenever X and Y are convergence vector spaces with Y Hausdorff, complete and topological. This is essentially known in the locally convex case [2], [3].

Indeed, if Y is locally convex, Hausdorff and complete, then Y is isomorphic to $\mathcal{L}_c\mathcal{L}_cY$, which is a closed subspace of $\mathcal{C}_c(\mathcal{L}_cY)$. Thus $\mathcal{L}_c(X, Y)$ is isomorphic to a closed subspace of $\mathcal{C}_c(X, \mathcal{C}_c(\mathcal{L}_cY))$. By the Universal Property of the continuous convergence structure, $\mathcal{C}_c(X, \mathcal{C}_c(\mathcal{L}_cY))$ is isomorphic to $\mathcal{C}_c(X \times \mathcal{L}_cY)$, which is complete [3]. Hence $\mathcal{L}_c(X, Y)$ is a closed subspace of a complete convergence vector space, and is therefore complete.

2. A completeness result

We now show that the following more result holds. This result generalizes [2, Theorem 3.1.15].

Theorem 2.1. *Let X be a convergence space and Y a Hausdorff, complete topological vector space. Then $\mathcal{C}_c(X, Y)$ is complete.*

Date: January 22, 2010.
2000 *Mathematics Subject Classification.* Primary 46A19, 46A32; Secondary 46E10.
Key words and phrases. Convergence space, Continuous convergence, Completeness.

Proof. Let Φ be a Cauchy filter on $\mathcal{C}_c(X,Y)$ so that

$$\begin{array}{ll} \forall & x \in X : \\ \forall & \mathcal{F} \in \lambda_X(x) : \\ & \omega_{X,Y}\left(\mathcal{F}, \Phi - \Phi\right) \in \lambda_Y\left(0\right) \end{array}, \tag{2.1}$$

where $\omega_{X,Y} : X \times \mathcal{C}(X,Y) \to Y$ is the evaluation mapping, defined through

$$\omega_{X,Y}\left(x, f\right) = f\left(x\right).$$

In particular, upon setting $\mathcal{F} = [x]$ in (2.1) we obtain

$$\Phi\left(x\right) - \Phi\left(x\right) = \Phi\left([x]\right) - \Phi\left([x]\right) = \omega_{X,Y}\left([x], \Phi - \Phi\right) \in \lambda_Y(0)$$

for every $x \in X$. Therefore $\Phi(x)$ is a Cauchy filter in Y for every $x \in X$. Since Y is complete and Hausdorff, it follows that

$$\begin{array}{ll} \forall & x \in X : \\ \exists! & x_\Phi \in Y : \\ & \Phi(x) \in \lambda_Y\left(x_\Phi\right) \end{array}.$$

Define the mapping $f : X \to Y$ through

$$f : X \ni x \mapsto x_\Phi \in Y. \tag{2.2}$$

We show that f is continuous. Note that, since Y is topological, there is a collection $\mathcal{B}$ of closed subsets of Y such that filter $\mathcal{G} = [\mathcal{B}]$ converges to 0 and

$$\begin{array}{ll} \forall & \mathcal{F} \in \lambda_Y(0) : \\ & \mathcal{G} \subseteq \mathcal{F} \end{array}. \tag{2.3}$$

Let $\mathcal{F}$ converge to $x_0 \in X$. Without loss of generality, we may assume that $\mathcal{F} \subseteq [x_0]$. Since the filter $\omega_{X,Y}\left(\mathcal{F}, \Phi - \Phi\right)$ converges to 0 in Y, it follows by (2.3) that $\mathcal{G} \subseteq \omega_{X,Y}\left(\mathcal{F}, \Phi - \Phi\right)$. We therefore have

$$\begin{array}{ll} \forall & B \in \mathcal{B} : \\ \exists & A_B \in \Phi : \\ \exists & F_B \in \mathcal{F} : \\ & \omega_{X,Y}\left(F_B, A_B - A_B\right) \subseteq B \end{array}$$

so that

$$\left(A_B - A_B\right)\left(F_B\right) = \left\{ g(x) - h(x) \,\middle|\, \begin{array}{l} g, h \in A_B \\ x \in F_B \end{array} \right\} \subseteq B.$$

In particular,

$$\begin{array}{ll} \forall & x \in F_B : \\ \forall & g \in A_B : \\ & A_B\left(x\right) = \{h(x) : h \in A_B\} \subseteq g\left(x\right) + B \end{array} \tag{2.4}$$

Since $f\left(x\right)$ is defined as the limit of $\Phi(x)$ in Y, it follows that $f\left(x\right) \in a_Y\left(A_B\left(x\right)\right)$, where a_Y denotes the adherence operator in Y. Since B is closed in Y, it follows from (2.4) that

$$\begin{array}{ll} \forall & g \in A_B : \\ \forall & x \in F_B : \\ & f\left(x\right) \in g\left(x\right) + B \end{array}. \tag{2.5}$$

For every $B \in \mathcal{B}$ and $A \in \Phi$, pick some $g \in A_B \cap A$. Since g is continuous, the filter $g(\mathcal{F})$ converges to $g(x_0)$ in Y. It now follows from (2.3) that $\mathcal{G} + g(x_0) \subseteq g(\mathcal{F})$. Fix $B \in \mathcal{B}$. Then

$$\begin{array}{ll} \exists & F_{B,0} \in \mathcal{F} : \\ & g(F_{B,0}) \subseteq B + g(x_0) \end{array}$$

so that (2.5) implies

$$f(F_B \cap F_{B,0}) \subseteq B + g(x_0) \subseteq (A_B \cap A)(x_0) + B \subseteq A(x_0) + B.$$

Since $B \in \mathcal{B}$ and $A \in \Phi$ were arbitrary, it follows that $\Phi(x_0) + \mathcal{G} \subseteq f(\mathcal{F})$. By definition, $\Phi(x_0)$ converges to $f(x_0)$, and since $\mathcal{G}$ converges to 0 it follows that $f(\mathcal{F})$ converges to $f(x_0)$ which shows that f is continuous.

Now we show that Φ converges continuously to f. Choose $x_0 \in X$ and $\mathcal{F} \in \lambda_X(x_0)$ as above so that we have

$$\begin{array}{ll} \forall & B \in \mathcal{B} : \\ \exists & A_B \in \Phi : \\ \exists & F_B \in \mathcal{F} : \\ & g \in A_B, x \in F_B \Rightarrow f(x) \in g(x) + B \end{array} . \tag{2.6}$$

Since f is continuous, we also have

$$\begin{array}{ll} \forall & B \in \mathcal{B} : \\ \exists & F_{B,0} \in \mathcal{F} : \\ & f(F_{B,0}) \subseteq f(x_0) + B \end{array} . \tag{2.7}$$

From (2.6) and (2.7) it follows that

$$\begin{array}{ll} \forall & x \in F_B \cap F_{B,0} : \\ & A_B(x) \subseteq f(x) - B \subseteq f(x_0) + B - B. \end{array} .$$

Therefore

$$\omega_{X,Y}(F_B \cap F_{B,0}, A_B) \subseteq f(x_0) + B - B.$$

Consequently, $[f(x_0)] + \mathcal{G} - \mathcal{G} \subseteq \omega_{X,Y}(\Phi, \mathcal{F})$ so that $\omega_{X,Y}(\Phi, \mathcal{F})$ converges to $f(x_0)$. Since $x_0 \in X$ was chosen arbitrary, it follows that Φ converges continuously to f. This completes the proof. □

Corollary 2.2. If X and Y are convergence vector spaces, Y Hausdorff, complete and topological, then $\mathcal{L}_c(X,Y)$ is complete.

Remark 2.3. It should be noted that the proof of Theorem 2.1 given here cannot be used in the case of a nontopological range space Y. Indeed, the proof depends heavily on the existence of a filter $\mathcal{G}$, with a basis of closed sets, which converges to 0 in Y and satisfies

$$\begin{array}{ll} \forall & \mathcal{F} \in \lambda_Y(0) : \\ & \mathcal{G} \subseteq \mathcal{F} \end{array} .$$

Clearly the existence of such a filter implies that Y is pretopological, and hence topological.

While the techniques used in the proof of Theorem 2.1 does not apply to nontopological spaces, similar arguments suffice if Y is replaced with a complete, Hausdorff and commutative topological group. In particular, the following is true.

Theorem 2.4. *Let X be a convergence space, and Y a complete, Hausdorff commutative topological group. Then $\mathcal{C}_c(X,Y)$ is a complete convergence group.*

Since the proof is based on almost exactly the same arguments used to verify Theorem 2.1 we do not give it here.

Lastly, we mention that the completeness result for $\mathcal{L}_c(X,Y)$, with Y a complete locally convex space, or more generally any continuously reflexive convergence vector space [2], mentioned earlier, has been used successfully in infinite dimensional analysis, see for instance [4] and [6]. Our results may therefore have a wide range of applicability in analysis on non locally convex spaces [1].

References

[1] A. Bayoumi, *Foundations of complex analysis in non locally convex spaces: Function theory without convexity conditions*, Elsevier, 2003.

[2] R. Beattie and H.-P. Butzmann, *Convergence structures and applications to functional analysis*, Kluwer Academic Plublishers, 2002.

[3] E. Binz, *Continuous convergence in $\mathcal{C}_c(X)$*, Lecture Notes in Mathematics **469**, Springer-Verlag, 1975.

[4] S. Bjon and M. Lindström, *A general approach to infinite-dimensional holomorphy*, Monatshefte für Mathematik **101** (1986), no. 1, 11–26.

[5] H.-P. Butzmann, *An incomplete function space*, Applied Categorical Structures **9** (2001), no. 4, 365–368.

[6] L. D. Nel, *Categorical differential calculus for infinite dimensional spaces*, Cahiers Topologie Géométrie Différentielle Catégoriques **29** (1988), no. 4, 257–286.

Department of Mathematics and Applied Mathematics, University of Pretoria, Pretoria 0002, South Africa

E-mail address: janharm.vanderwalt@up.ac.za